J Glob Optim (2010) 46:383–393
DOI 10.1007/s10898-009-9431-1

Hadamard well-posed vector optimization problems

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Received: 30 November 2008 / Accepted: 30 April 2009 / Published online: 15 May 2009

Abstract In this paper, two kinds of Hadamard well-posedness for vector-valued optimization problems are introduced. By virtue of scalarization functions, the scalarization theorems of convergence for sequences of vector-valued functions are established. Then, sufficient conditions of Hadamard well-posedness for vector optimization problems are obtained by using the scalarization theorems.

Keywords Vector optimization · Variational convergence · Γ_C-convergence · Hadamard well-posedness

1 Introduction

For well-posed optimization problems, there are concepts of two main types: Tykhonov well-posedness and Hadamard well-posedness. In 1966, Tykhonov [16] first introduced a concept of well-posedness imposing convergence of every minimizing sequence to the unique minimum point, which is called Tykhonov well-posedness. In the last decades, some extensions of this concept for vector optimization problems appeared, see [2,3,6,7,12] and the references therein. Loridan [9] gave a survey on some theoretical results of well-posedness, approximate solutions and variational principles in vector optimization. Based on the ε-minimal solutions, Bednarczuk [2] investigated several Tykhonov types of well-posedness for vector optimization problems. Huang [6] introduced three kinds of extended Tykhonov well-posedness properties for vector-valued optimization problems and investigated a series of

This research was partially supported by the National Natural Science Foundation of China (grant numbers: 10871216 and 60574073).

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their characterizations and criteria. Miglierina et al. [12] listed and classified some existing notions of Tykhonov well-posedness for vector optimization problems and compared them. The concept of Hadamard well-posedness is inspired by the classical idea of Hadamard, which goes back to the beginning of the last century. It requires existence and uniqueness of the optimal solution together with continuous dependence on the problem data. So, Hadamard well-posedness is deeply linked with stability of vector optimization problems. Luc [10] addressed stability properties of solution sets of vector problems in his book. Lucchetti and Miglierina [11] investigated that sets of minimal points of the images of perturbed problems converge to the set of the minimal points of the original problem.

In this paper, we further investigate Hadamard well-posedness for vector-valued optimization problems. By using the definition of variational convergence for vector-valued sequences of functions introduced by Oppezzi and Rossi [14] very recently, we define two different notions of Hadamard well-posedness for vector-valued optimization problems, i.e., extended Hadamard well-posedness and generalized Hadamard well-posedness. Moreover, we show scalarization theorems of convergence for sequences of vector-valued functions by using the scalarization results introduced in [5]. Finally, based on scalarization theorems we derived, we extend some basic results of Hadamard well-posedness of scalar optimization problems to the case of vector-valued optimization problems, and then get sufficient conditions for Hadamard well-posedness of vector-valued optimization problems.

The paper is organized as follows. In Sect. 2, we present the concepts of two kinds of Hadamard well-posedness for vector-valued optimization problems and give examples to illustrate them. In Sect. 3, we prove scalarization theorems for convergence of sequences of vector-valued functions. In Sect. 4, we extend Hadamard well-posedness results of scalar optimization problems to those of vector-valued optimization problems.

2 Preliminaries and notations

Let X be a topological vector space and Y be a topological vector space ordered by a convex closed and pointed cone $C \subset Y$ with its topological interior int$C \neq \emptyset$. For $y, y' \in Y$, we write $y \leq y'$ if $y' - y \in C$.

Let us consider the scalar-valued functions $I_n, I : X \to [-\infty, +\infty]$.

Definition 2.1 [4] We say that I_n converges variationally to I, and write var-lim$I_n = I$, iff $x_n \to x$ implies $\liminf_n I_n(x_n) \geq I(x)$ and for every $u \in X$ there exists $u_n \in X$ such that $\limsup_n I_n(u_n) \leq I(u)$.

Proposition 2.1 *If $I_n, I : X \to [-\infty, +\infty]$ satisfy that for every $x \in X$,*

$$\sup_{U \in \mathcal{U}(x)} \limsup_n \inf I_n(U) \leq I(x) \leq \sup_{U \in \mathcal{U}(x)} \liminf_n \inf I_n(U), \tag{1}$$

(where $\mathcal{U}(x)$ is the system of neighborhoods of x), then var-lim$I_n = I$.

Proof (i) Assume that $x_n \to x$. If $I(x) \in R \cup \{+\infty\}$, the second inequality in (1) implies that for arbitrarily chosen $\varepsilon > 0$, $\exists U_\varepsilon \in \mathcal{U}(x)$ such that $\liminf_n \inf I_n(U_\varepsilon) \geq I(x) - \varepsilon$. From $x_n \to x$, there exists $k_\varepsilon > 0$ such that $\forall n \geq k_\varepsilon$, $x_n \in U_\varepsilon$. Then, $\liminf_n I_n(x_n) \geq \liminf_n \inf I_n(U_\varepsilon) \geq I(x) - \varepsilon$. Therefore, $\liminf_n I_n(x_n) \geq I(x)$. On the other hand, it is obvious that $\liminf_n I_n(x_n) \geq I(x)$ if $I(x) = -\infty$.

(ii) $\forall u \in X$, if $I(u) \in R \cup \{-\infty\}$, the first inequality in (1) implies that, $\forall U \in \mathcal{U}(u)$, we have $\limsup_n \inf I_n(U) \leq I(u)$. Noticing that for arbitrarily chosen $\varepsilon_n > 0$, there exists $u_n \in X$ such that $I_n(u_n) \leq \inf I_n(U) + \varepsilon_n$. Therefore, $\limsup_n I_n(u_n) \leq \limsup_n \inf I_n(U) \leq I(u)$. On the other hand, if $I(u) = +\infty$, it is obvious that $\limsup_n I_n(u_n) \leq I(u)$ for arbitrarily chosen $u \in X$. □

In [14], the following definition of convergence for vector-valued functions is introduced, which is a generalization of Definition 2.1.

Definition 2.2 [14] Let $\mathcal{U}(x)$ be the family of neighborhoods of $x \in X$, $f_n, f : X \to Y (n \in N)$ be given functions. We say that $(f_n)_{n \in N}\Gamma_C$-converges to f and we shall write $f_n \overset{\Gamma_C}{\to} f$, if for every $x \in X$:

(i) $\forall U \in \mathcal{U}(x), \forall q_0 \in intC, \exists n_{q_0,U} \in N$ such that $\forall n \geq n_{q_0,U}$, $\exists x_n \in U$ such that $f_n(x_n) \leq f(x) + q_0$;
(ii) $\forall q_0 \in intC, \exists U_{q_0} \in \mathcal{U}(x), k_{q_0} \in N$ such that $f_n(x') \geq f(x) - q_0$, $\forall x' \in U_{q_0}$, $\forall n \geq k_{q_0}$.

Definition 2.3 [13, Definition 4.1] We say that $f : X \to Y$ is strongly lower (upper) C-semicontinuous at the point $x_0 \in X$ if for any $q_0 \in \text{int}C$ there exists U_{x_0,q_0}, a neighborhood of x_0, such that $\forall x \in U_{x_0,q_0}$, we have $f(x) \in f(x_0) - q_0 + \text{int}C (f(x_0) \in f(x) - q_0 + \text{int}C)$.

Remark 2.1 Suppose that $Y = R, C = R_+$ and $q_0 = 1$. Then the strongly lower (upper) C-semicontinuity of $f : X \to Y$ reduces to lower (upper) semi-continuity in the scalar sense.

Lemma 2.1 [14, Proposition 2.6] *Let $f_n, f : X \to Y, n \in N$. If $f_n \overset{\Gamma_C}{\to} f$, then f is strongly lower C-semicontinuous.*

Consider the following vector-valued optimization problem:

$$(S, f) : \min_{x \in S} f(x),$$

where $f : S \to Y$ and S is a nonempty subset of X. Let us recall that x_0 is an efficient solution (resp. weak efficient solution) for problem (S, f) if $(f(x_0) - C \setminus \{0\}) \bigcap f(S) = \emptyset$ (resp. $(f(x_0) - \text{int}C) \bigcap f(S) = \emptyset$). The set of efficient solutions (resp. weak efficient solutions) to problem (S, f) is denoted by $\text{Eff}(f, S, C)$(resp. $\text{WEff}(f, S, C)$). If $Y = R$ and $C = R_+$, then (S, f) is a scalar optimization problem. We denote the solution set for the scalar optimization problem by $\text{Inf}(f, S)$ and we denote the minimizing value of the scalar optimization problem by $\text{val}(S, f)$.

Let us consider $Y = R$ and $C = R_+$. It is said that x_0 is an approximate solution for the scalar problem (S, f) if $f(x_0) - \varepsilon \leq f(x), \forall x \in S$. The set of approximate solutions for the scalar problem (S, f) is denoted by $\text{Inf}(f, S, \varepsilon)$. This notion can be extended to vector optimization problems by the following definition, which is introduced by Kutateladze [8].

Definition 2.4 [8] Let us consider $q \in \text{int}C$, $\varepsilon \geq 0$. It is said that x_0 is an εq-efficient solution (resp. weak εq-efficient solution) for problem (S, f) if

$$(f(x_0) - \varepsilon q - C \setminus \{0\}) \bigcap f(S) = \emptyset$$
$$\text{resp.}\ \ (f(x_0) - \varepsilon q - \text{int}C) \bigcap f(S) = \emptyset.$$

The set of εq-efficient solutions (resp. weak εq-efficient solutions) is denoted by $\text{Eff}(f, S, C, \varepsilon q)$ (resp. $\text{WEff}(f, S, C, \varepsilon q)$). It is obvious that $\text{Eff}(f, S, C, 0q) = \text{Eff}(f, S, C)$ [resp. $\text{WEff}(f, S, C, 0q) = \text{WEff}(f, S, C)$].

Assume that $f : S \to Y$, $q \in \mathrm{int}C$ and for all $n \in N$, $f_n : S \to Y$. Let $\{A_n\}$ be a sequence of subsets of X. It is said that $z \in \mathrm{Limsup}_n A_n$ (outer limit of $\{A_n\}$ in [15]) if, there exist a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ and a sequence $\{z_{n_k}\}$ converging to z such that $z_{n_k} \in A_{n_k}$ for each $n_k \in N$.

Now we introduce two notions of Hadamard well-posedness for vector optimization problems.

Definition 2.5 Let $f_n \overset{\Gamma_C}{\to} f$. (S, f) is said to be generalized Hadamard well-posed with respect to $\{f_n\}$, if $\mathrm{Limsup}_n[\mathrm{WEff}(f_n, S, C, \varepsilon_n q)] \subset \mathrm{WEff}(f, S, C)$, for $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$.

Definition 2.6 Let $f_n \overset{\Gamma_C}{\to} f$. (S, f) is said to be extended Hadamard well-posed with respect to $\{f_n\}$, if there exists $\varepsilon_0 > 0$ such that $\mathrm{Limsup}_n[\mathrm{WEff}(f_n, S, C, \varepsilon q)] \subset \mathrm{WEff}(f, S, C, \varepsilon q)$, for all $0 \leq \varepsilon \leq \varepsilon_0$.

Remark 2.2 (a) Suppose that $Y = R$, $C = R_+$ and $q = 1$. If $f_n = f$ for every n, then generalized Hadamard well-posedness with respect to $\{f_n\}$ coincides with Tykhonov well-posedness in the generalized sense [4, Chap. I, Sect. 6]. If $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$, $f_n = f - \varepsilon_n$ and f is lower-semicontinuous, then extended Hadamard well-posedness with respect to $\{f_n\}$ also coincides with Tykhonov well-posedness in the generalized sense [4, Chap. I, Sect. 6].

(b) If (S, f) is generalized Hadamard well-posed with respect to $\{f_n\}$, then

$$\mathrm{Limsup}_n \mathrm{WEff}(f_n, S, C) \subset \mathrm{WEff}(f, S, C).$$

Let us illustrate these definitions by the following examples.

Example 2.1 Let $X = R$, $Y = R^2$, $C = R^2_+$ and $q = (1, 1)$.

(i) Let $S = R$, $f_n : S \to R^2$ be defined for every $n \in N$ and $x \in R$ by

$$f_n(x) = \begin{cases} (x, 0), & \text{if } x \leq 0, \\ (x, nx), & \text{if } 0 \leq x \leq \dfrac{1}{n}, \\ (x, 1), & \text{if } x \geq \dfrac{1}{n}. \end{cases}$$

We can easily verify that $f_n \overset{\Gamma_C}{\to} f$ with

$$f(x) = \begin{cases} (x, 0), & \text{if } x \leq 0, \\ (x, 1), & \text{if } x > 0. \end{cases}$$

It is easy to get that (S, f) is generalized Hadamard well-posed with respect to $\{f_n\}$ but not extended Hadamard well-posed with respect to $\{f_n\}$.

(ii) Let $S = R$, $f_n : S \to R^2$ be defined for every $n \in N$ and $x \in R$ by

$$f_n(x) = \begin{cases} (x, x), & \text{if } x \geq 0, \\ \frac{1}{n}(x, x), & \text{if } 0 \geq x \geq -n, \\ (-1, -1), & \text{if } -n \geq x. \end{cases}$$

We can easily verify that $f_n \overset{\Gamma_C}{\to} f$ with

$$f(x) = \begin{cases} (x, x), & \text{if } x \geq 0, \\ (0, 0), & \text{if } x \leq 0. \end{cases}$$

Then, $\forall \varepsilon_n \to 0$, $\varepsilon_n \geq 0$, $\text{WEff}(f_n, S, C, \varepsilon_n q) = (-\infty, -n(1 - \varepsilon_n)]$. We obtain $\text{Limsup}_n(\text{WEff}(f_n, S, C, \varepsilon_n q)) = \emptyset$, which is included in $\text{WEff}(f, S, C) = (-\infty, 0]$. Moreover, $\text{WEff}(f_n, S, C, \varepsilon q) = (-\infty, -n(1 - \varepsilon)]$, $\forall \varepsilon < 1$, and $\text{WEff}(f, S, C, \varepsilon q) = (-\infty, \varepsilon] \supset \text{Limsup}_n(\text{WEff}(f_n, S, C, \varepsilon q))$. Therefore, (S, f) is both extended Hadamard well-posed with respect to $\{f_n\}$ and generalized Hadamard well-posed with respect to $\{f_n\}$.

Proposition 2.2 *Let $f_n, f : S \to Y$, $f_n \xrightarrow{\Gamma_C} f$. If (S, f) is extended Hadamard well-posed with respect to $\{f_n\}$, then it is generalized Hadamard well-posed with respect to $\{f_n\}$.*

Proof Assume that the problem (S, f) is not generalized Hadamard well-posed with respect to $\{f_n\}$. Hence, for arbitrarily chosen $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$, there exists $\bar{x}$ satisfying

$$\bar{x} \in \text{Limsup}_n[\text{WEff}(f_n, S, C, \varepsilon_n q)] \tag{2}$$

and $\bar{x} \notin \text{WEff}(f, S, C)$. From (2), there exist $x_n \in \text{WEff}(f_n, S, C, \varepsilon_n q)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \to \bar{x}. \tag{3}$$

Since $\bar{x} \notin \text{WEff}(f, S, C)$, we have that $\forall \varepsilon_1 > 0$, there exists $0 < \varepsilon_2 < \varepsilon_1$ satisfying

$$\bar{x} \notin \text{WEff}(f, S, C, \varepsilon_2 q). \tag{4}$$

In fact, if not, $\exists \varepsilon_1 > 0$, for all $0 < \varepsilon < \varepsilon_1$, such that $\bar{x} \in \text{WEff}(f, S, C, \varepsilon q)$. It is said that $(f(\bar{x}) - \varepsilon q - \text{int}C) \cap f(S) = \emptyset$, i.e., $(f(\bar{x}) - \varepsilon q) \in X \backslash (f(S) + \text{int}C)$. Let $\varepsilon \to 0$, we have that $f(\bar{x}) \in X \backslash (f(S) + \text{int}C)$ by the closeness of $X \backslash (f(S) + \text{int}C)$. Thus, $(f(\bar{x}) - \text{int}C) \cap f(S) = \emptyset$, which contradicts to $\bar{x} \notin \text{WEff}(f, S, C)$.

Moreover, for a fixed $\varepsilon' > 0$ and sequence $\{\varepsilon_n\}$ satisfying $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$, there exists $N \geq 0$, such that $\forall n \geq N$,

$$\text{WEff}(f_n, S, C, \varepsilon_n q) \subset \text{WEff}(f_n, S, C, \varepsilon' q). \tag{5}$$

Indeed, for ε', there exists $N \geq 0$, such that $\forall n \geq N$, $\varepsilon_n \leq \varepsilon'$. $\forall x \in \text{WEff}(f_n, S, C, \varepsilon_n q)$, we have $(f(x) - \varepsilon_n q - \text{int}C) \cap f_n(S) = \emptyset$. According to $-\varepsilon' q - \text{int}C \subset -\varepsilon_n q - \text{int}C$, we conclude $x \in \text{WEff}(f_n, S, C, \varepsilon' q)$.

From (4) and (5), we get that $\forall \varepsilon_1 > 0$, there exist $0 < \varepsilon_2 < \varepsilon_1$ and $N \geq 0$ such that $x_n \in \text{WEff}(f_n, S, C, \varepsilon_n q) \subset \text{WEff}(f_n, S, C, \varepsilon_2 q)$ for all $n \geq N$ and $\bar{x} \notin \text{WEff}(f, S, C, \varepsilon_2 q)$. Thus, from Definition 2.6 and (3), (f, S) is not extended Hadamard well-posed with respect to $\{f_n\}$, which is a contradiction. □

3 Scalarization of variational convergence for vector-valued sequences of functions

In this section, we scalarize vector-valued sequences of mappings, and show that the scalarized sequences of vector-valued mappings are variational converging when the sequences of vector-valued mappings are Γ_C-converging.

Proposition 3.1 *Suppose that $f_n, f : X \to Y$, $f_n \xrightarrow{\Gamma_C} f$, and the scalarization functional $g : Y \to [-\infty, +\infty]$ satisfying $g(q) \to 0$ when $q \to 0$. Moreover, assume that g is monotone ($i.e. \forall y_1, y_2 \in Y, y_1 \leq y_2$ implies $g(y_1) \leq g(y_2)$), sub-additive ($i.e. \forall y_1, y_2 \in Y$, $g(y_1 + y_2) \leq g(y_1) + g(y_2)$). Then var-lim$g \circ f_n = g \circ f$.*

Proof From Definition 2.2 (i), $\forall U \in \mathcal{U}(x), \forall q \in intC, \exists n_{q,U} \in N$ such that $\forall n \geq n_{q,U}$, $\exists x_n \in U$ satisfying

$$f_n(x_n) \leq f(x) + q.$$

Since g is monotone and sub-additive, $g \circ f_n(x_n) \leq g \circ f(x) + g(q)$. This implies that $\inf g \circ f_n(U) \leq g \circ f(x) + g(q)$. Hence, $\limsup_n \inf g \circ f_n(U) \leq g \circ f(x) + g(q)$. Therefore, $\sup_{U \in \mathcal{U}(x)} \limsup_n \inf g \circ f_n(U) \leq g \circ f(x) + g(q)$. Let $q \to 0$, we have

$$\sup_{U \in \mathcal{U}(x)} \limsup_n \inf g \circ f_n(U) \leq g \circ f(x). \tag{6}$$

From Definition 2.2 (ii), $\forall q \in intC, \exists U_q \in \mathcal{U}(x), k_q \in N$ such that

$$f_n(x') \geq f(x) - q, \quad \forall x' \in U_q, \quad \forall n \geq k_q.$$

From the properties of g, we have $g \circ f_n(x') \geq g \circ f(x) - g(q)$. Therefore, $\inf g \circ f_n(U_q) \geq g \circ f(x) - g(q)$, so $\liminf_n \inf g \circ f_n(U_q) \geq g \circ f(x) - g(q)$. Let $q \to 0$, we have $\liminf_n \inf g \circ f_n(U_q) \geq g \circ f(x)$. This implies that

$$\sup_{U_q \in \mathcal{U}(x)} \liminf_n \inf g \circ f_n(U_q) \geq g \circ f(x). \tag{7}$$

From (6) and (7), we get var-lim$g \circ f_n = g \circ f$. □

According to [5], for fixed $q \in \mathrm{int}C$, $f : X \to Y$ and for all $x_0 \in X, \varepsilon \geq 0$, the scalarization functional $\varphi_{x_0,\varepsilon} : Y \to [-\infty, +\infty]$ is defined by

$$\varphi_{x_0,\varepsilon}(y) = \inf\{s \in R : y \in sq + f(x_0) - \varepsilon q - C\}, \quad \forall y \in Y.$$

Lemma 3.1 [5, Lemma 4.4.] *For all $x_0 \in X$, $\varepsilon \geq 0$, we have*

(i) $\varphi_{x_0,\varepsilon}(\cdot)$ *is a continuous, convex and strictly monotone functional satisfying*

$$\{y \in Y : \varphi_{x_0,\varepsilon}(y) < 0\} = f(x_0) - \varepsilon q - intC; \tag{8}$$

(ii) $\varphi_{x_0,\varepsilon}(f(x_0) + \rho q) = \varepsilon + \rho, \forall \rho \in R$;
(iii) $\varphi_{x_0,\varepsilon}(y) - \varphi_{x_0,\varepsilon}(y - \rho q) = \rho, \forall y \in Y, \forall \rho \in R$.

Theorem 3.1 *Assume that $f_n, f : X \to Y, x_n \to \bar{x}, f_n \overset{\Gamma_C}{\to} f$ and f is strongly upper C-semicontinuous. Then*

(i) $\forall \varepsilon \geq 0$, *var-lim* $\varphi_{x_n,\varepsilon} \circ f_n = \varphi_{\bar{x},\varepsilon} \circ f$.
(ii) $\forall \varepsilon_n \geq 0, \varepsilon_n \to 0$, *var-lim* $\varphi_{x_n,\varepsilon_n} \circ f_n = \varphi_{\bar{x},0} \circ f$.

Proof (I) (i) Since $f_n \overset{\Gamma_C}{\to} f, \forall U \in \mathcal{U}(y), \forall \rho > 0, \exists n_{\rho,U} \in N$ such that $\forall n \geq n_{\rho,U}, \exists y_n \in U$ satisfying $f_n(y_n) \leq f(y) + \rho q$. From Lemma 3.1 (i) and Remark 3.2 (ii) of [5], for any $\epsilon \geq 0$, $\varphi_{\bar{x},\varepsilon}$ is monotone. Together with Lemma 3.1 (iii), we have

$$\varphi_{\bar{x},\varepsilon} \circ f_n(y_n) \leq \varphi_{\bar{x},\varepsilon} \circ f(y) + \rho. \tag{9}$$

From Lemma 2.1, f is strongly lower C-semicontinuous. Then, $\forall \delta > 0, \exists U_{\bar{x},\delta}$ such that $\forall x \in U_{\bar{x},\delta}$, we have $f(\bar{x}) - \delta q \in f(x) - C$. So there exists $n_\delta \geq n_{\rho,U}$

such that $\forall n \geq n_\delta$, $f(\bar{x}) - \delta q \in f(x_n) - C$. It implies that $\forall \varepsilon \geq 0$, $sq + f(\bar{x}) - \delta q - \varepsilon q - C \subset sq + f(x_n) - \varepsilon q - C$. Therefore,

$$\begin{aligned}\varphi_{x_n,\varepsilon}(f_n(y_n)) &= \inf\{s \in R \mid f_n(y_n) \in sq + f(x_n) - \varepsilon q - C\}\\ &\leq \inf\{s \in R \mid f_n(y_n) \in sq + f(\bar{x}) - \delta q - \varepsilon q - C\}\\ &= \inf\{s \in R \mid f_n(y_n) \in sq + f(\bar{x}) - \varepsilon q - C\} + \delta\\ &= \varphi_{\bar{x},\varepsilon}(f_n(y_n)) + \delta.\end{aligned}$$

From (9), we have $\varphi_{x_n,\varepsilon} \circ f_n(y_n) \leq \varphi_{\bar{x},\varepsilon} \circ f(y) + \delta + \rho$, $\forall n \geq n_\delta$.
Thus, $\forall n \geq n_\delta$, $\inf \varphi_{x_n,\varepsilon} \circ f_n(U) \leq \varphi_{\bar{x},\varepsilon} \circ f(y) + \delta + \rho$, so $\sup_{U \in \mathcal{U}(y)} \limsup_n \inf \varphi_{x_n,\varepsilon} \circ f_n(U) \leq \varphi_{\bar{x},\varepsilon} \circ f(y) + \delta + \rho$. By the arbitrariness of δ and ρ, we obtain

$$\sup_{U \in \mathcal{U}(y)} \limsup_n \inf \varphi_{x_n,\varepsilon} \circ f_n(U) \leq \varphi_{\bar{x},\varepsilon} \circ f(y).$$

(ii) Since f is strongly lower C-semicontinuous, $\forall y \in S, \rho > 0, \exists U_\rho \in \mathcal{U}(y)$, $k_\rho \in N$ such that

$$f_n(y') \geq f(y) - \rho q, \quad \forall y' \in U_\rho, \quad \forall n \geq k_\rho.$$

From Lemma 3.1 (i) and (iii), we have

$$\varphi_{\bar{x},\varepsilon}(f_n(y')) \geq \varphi_{\bar{x},\varepsilon}(f(y)) - \rho. \tag{10}$$

From the strongly upper C-semicontinuity of f, $\forall \delta > 0$, there exists $U'_\delta \in \mathcal{U}(\bar{x})$ such that $\forall x' \in U'_\delta$, $f(\bar{x}) \in f(x') - \delta q + C$. It implies that there exists $k_{\delta,\rho} \geq k_\rho$ such that $\forall n \geq k_{\delta,\rho}$ we have $f(x_n) - \varepsilon q - C \subset f(\bar{x}) + \delta q - C - \varepsilon q$. Therefore,

$$\begin{aligned}\varphi_{x_n,\varepsilon}(f_n(y')) &= \inf\{s \in R \mid f_n(y') \in sq + f(x_n) - \varepsilon q - C\}\\ &\geq \inf\{s \in R \mid f_n(y') \in sq + f(\bar{x}) - \varepsilon q - C + \delta q\}\\ &= \inf\{s \in R \mid f_n(y') \in sq + f(\bar{x}) - \varepsilon q - C\} - \delta\\ &= \varphi_{\bar{x},\varepsilon}(f_n(y')) - \delta.\end{aligned}$$

From (10), we obtain that $\forall \rho, \delta > 0$, $\exists U_\rho \in \mathcal{U}(y)$, $k_{\delta,\rho} \in N$ such that

$$\varphi_{x_n,\varepsilon} \circ f_n(y') \geq \varphi_{\bar{x},\varepsilon} \circ f(y) - \rho - \delta, \forall y' \in U_\rho, \forall n \geq k_{\delta,\rho}. \tag{11}$$

Thus, $\inf \varphi_{x_n,\varepsilon} \circ f_n(U_\rho) \geq \varphi_{\bar{x},\varepsilon} \circ f(y) - \rho - \delta$. It is said that

$$\sup_{U_\rho \in \mathcal{U}(y)} \liminf_n \inf \varphi_{x_n,\varepsilon} \circ f_n(U_\rho) \geq \varphi_{\bar{x},\varepsilon} \circ f(y) - \rho - \delta.$$

By the arbitrariness of ρ and δ, we get $\sup_{U_\rho \in \mathcal{U}(y)} \liminf_n \inf \varphi_{x_n,\varepsilon} \circ f_n(U_\rho) \geq \varphi_{\bar{x},\varepsilon} \circ f(y)$.

From Proposition 2.1, $\forall \varepsilon \geq 0$, var-lim$\varphi_{x_n,\varepsilon} \circ f_n = \varphi_{\bar{x},\varepsilon} \circ f$.

(II) It is noticed that for arbitrarily chosen $x \in X$, $z \in Y$ and $\varepsilon \geq 0$, $\varphi_{x,\varepsilon}(z) = \varphi_{x,0}(z) + \varepsilon$. Thus, $\forall \varepsilon_n > 0$, $\varepsilon_n \to 0$, we have

$$\begin{aligned}\sup_{U_\rho \in \mathcal{U}(x)} \liminf_n \inf \varphi_{x_n,\varepsilon_n} \circ f_n(U_\rho) &= \sup_{U_\rho \in \mathcal{U}(x)} \liminf_n (\inf \varphi_{x_n,0} \circ f_n(U_\rho) + \varepsilon_n)\\ &= \sup_{U_\rho \in \mathcal{U}(x)} \liminf_n \inf \varphi_{x_n,0} \circ f_n(U_\rho).\end{aligned}$$

Similarly, $\sup_{U\in\mathcal{U}(x)} \limsup_n \inf \varphi_{x_n,\varepsilon_n}\circ f_n(U) = \sup_{U\in\mathcal{U}(x)} \limsup_n \inf \varphi_{x_n,0}\circ f_n(U)$. From (I), we obtain

$$\sup_{U\in\mathcal{U}(x)} \limsup_n \inf \varphi_{x_n,0} \circ f_n(U) \le \varphi_{\bar{x},0} \circ f(x) \le \sup_{U_\rho\in\mathcal{U}(x)} \liminf_n \inf \varphi_{x_n,0} \circ f_n(U_\rho).$$

Therefore,

$$\sup_{U\in\mathcal{U}(x)} \limsup_n \inf \varphi_{x_n,\varepsilon_n} \circ f_n(U) \le \varphi_{\bar{x},0} \circ f(x) \le \sup_{U_\rho\in\mathcal{U}(x)} \liminf_n \inf \varphi_{x_n,\varepsilon_n} \circ f_n(U_\rho).$$

By Proposition 2.1, $\forall\varepsilon_n \ge 0,\ \varepsilon_n \to 0,$ var-lim $\varphi_{x_n,\varepsilon_n} \circ f_n = \varphi_{\bar{x},0} \circ f$. □

4 Hadamard well-posedness properties of vector optimization problems

In this section, we extend some basic results of Hadamard well-posedness of scalar optimization problems to the cases of vector-valued optimization problems and then get sufficient conditions for Hadamard well-posedness of vector-valued optimization problems.

From Theorem 5 in Chapter 4 of [4], we have the following lemma.

Lemma 4.1 [4] *Assume that var-lim $I_n = I$. Then*

(i) $\limsup$ *val*$(S, I_n) \le$ *val*(S, I)*;*
(ii) $\limsup_n$[*Inf*$(I_n, S, \varepsilon)] \subset$ *Inf*(I, S, ε) *for all sufficiently small* $\varepsilon \ge 0$*;*
(iii) *if* $\varepsilon_n \ge 0,\ \varepsilon_n \to 0$*, then* $\limsup_n$[*Inf*$(I_n, S, \varepsilon_n)] \subset$ *Inf*(I, S).

Lemma 4.2 [5, Theorem 5.2] *Assume that $f : S \to Y$ and $\varepsilon \ge 0$. Then $x_0 \in$ WEff$(f, S, C, \varepsilon q) \Leftrightarrow x_0 \in$ Inf$(\varphi_{x_0,\varepsilon} \circ f, S, \varepsilon)$.*

Theorem 4.1 *Assume that $f_n, f : S \to Y,\ f_n \overset{\Gamma_C}{\to} f$ and f is strongly upper C-semicontinuous. Then*

(i) $\forall x_n \to \bar{x},\ \forall\varepsilon_n \ge 0,\ \varepsilon_n \to 0,\ \limsup_n$ *val*$(S, \varphi_{x_n,\varepsilon_n} \circ f_n) \le$ *val*$(S, \varphi_{\bar{x},0} \circ f)$*, and for arbitrarily chosen* $\varepsilon \ge 0,\ \limsup_n$ *val*$(S, \varphi_{x_n,\varepsilon} \circ f_n) \le$ *val*$(S, \varphi_{\bar{x},\varepsilon} \circ f)$*;*
(ii) (S, f) *is extended Hadamard well-posed with respect to* $\{f_n\}$.

Proof The proof of (i) is clear. We only need to prove (ii).

Let $\bar{x} \in$ Limsup$_n$[WEff$(f_n, S, C, \varepsilon q)$], i.e. $\exists\{n_k\} \subset N,\ x_{n_k} \in$ WEff$(f_{n_k}, S, C, \varepsilon q)$ such that $x_{n_k} \to \bar{x}$. From Lemma 4.2, $x_{n_k} \in$ Inf$(\varphi_{x_{n_k},\varepsilon} \circ f_{n_k}, S, \varepsilon)$. Therefore, $\bar{x} \in$ Limsup$_{n_k}$[Inf$(\varphi_{x_{n_k},\varepsilon} \circ f_{n_k}, S, \varepsilon)$]. By Theorem 3.1(I), we have var-lim$\varphi_{x_{n_k},\varepsilon} \circ f_{n_k} = \varphi_{\bar{x},\varepsilon} \circ f$. From Lemma 4.1, it can be deduced that there exists $\varepsilon_0 > 0$ such that

$$\text{Limsup}_{n_k}[\text{Inf}(\varphi_{x_{n_k},\varepsilon} \circ f_{n_k}, S, \varepsilon)] \subset \text{Inf}(\varphi_{\bar{x},\varepsilon} \circ f, S, \varepsilon),\ \forall 0 \le \varepsilon \le \varepsilon_0.$$

It follows that $\bar{x} \in$ Inf$(\varphi_{\bar{x},\varepsilon} \circ f, S, \varepsilon)$. By Lemma 4.2, $\bar{x} \in$ WEff$(f, S, C, \varepsilon q)$.

Therefore, there exists $\varepsilon_0 > 0$ such that $\forall 0 \le \varepsilon \le \varepsilon_0$,

$$\text{Limsup}_n[\text{WEff}(f_n, S, C, \varepsilon q)] \subset \text{WEff}(f, S, C, \varepsilon q).$$

We conclude that (S, f) is extended Hadamard well-posed with respect to $\{f_n\}$. □

Remark 4.1 (a) From Theorem 4.1(ii) and Proposition 2.2, if the conditions of Theorem 4.1 hold, the problem (S, f) is generalized well-posedness with respect to $\{f_n\}$.

(b) The following example shows that without the assumption of strongly upper C-semicontinuity of f, conclusions of Theorem 4.1 may not hold. Assume that $f_n, f : R \to R^2$ defined as $f_n(x) = (x, nxe^{-2n^2x^2})$ for any $n \in N$, and

$$f(x) = \begin{cases} (x, 0), & \text{if} \quad x \neq 0, \\ (0, -\frac{1}{2}e^{-1/2}), & \text{if} \quad x = 0, \end{cases}$$

respectively. Now we show that $f_n \overset{\Gamma_C}{\to} f$.
In fact, if $x \neq 0$, we notice that $nx_n e^{-2n^2x_n^2} \to 0$ when $x_n \to x$. Then, we have that $\forall x_n \to x$, $\forall U \in \mathcal{U}(x), \forall q_0 \in intC, \exists n_{q_0,U} \in N$ such that $\forall n \geq n_{q_0,U}$,

$$(x_n, nx_n e^{-2n^2x_n^2}) \leq (x, 0) + q_0. \tag{12}$$

Moreover, we have that $\forall q_0 \in intC, \exists U_{q_0} \in \mathcal{U}(x), k_{q_0} \in N$ such that $\forall x' \in U_{q_0}, \forall n \geq k_{q_0}$,

$$\left(x', nx'e^{-2n^2x'^2}\right) \geq (x, 0) - q_0. \tag{13}$$

If $x = 0$, by taking $x_n = -\frac{1}{2n}$, we have $\forall U \in \mathcal{U}(x), \forall q_0 \in intC, \exists n'_{q_0,U} \in N$ such that $\forall n \geq n'_{q_0,U}$,

$$(x_n, nx_n e^{-2n^2x_n^2}) = \left(-\frac{1}{2n}, -\frac{1}{2}e^{-\frac{1}{2}}\right) \leq \left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right) + q_0. \tag{14}$$

And since $nxe^{-2n^2x^2} \geq -\frac{1}{2}e^{-\frac{1}{2}}$ for all $x \in R$, we have that $\forall q_0 \in intC, \exists U'_{q_0} \in \mathcal{U}(0), k'_{q_0} \in N$ such that $\forall x' \in U'_{q_0}, \forall n \geq k'_{q_0}$,

$$(x', nx'e^{-2n^2x'^2}) \geq \left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right) - q_0. \tag{15}$$

Therefore, it follows from (12), (13), (14), (15) and Definition 2.2 that $f_n \overset{\Gamma_C}{\to} f$.
However, because

$$\text{WEff}(f, S, C) = \left\{\left(0, -\frac{1}{2}e^{-1/2}\right)\right\}$$

and

$$\text{WEff}(f_n, S, C) = \left\{\left(x, nxe^{-2n^2x^2}\right) \mid x \leq -\frac{1}{2n}\right\}.$$

We have $\text{Limsup}_n[\text{WEff}(f_n, S, C)] \not\subset \text{WEff}(f, S, C)$. It is said that (S, f) is not extended well-posed with respect to $\{f_n\}$.

(c) We use the following example to illustrate Theorem 4.1. Let $S = R$, $C = R^2_+$ and $f_n : S \to R^2$ be defined for every $n \in N$ and $x \in R$ by

$$f_n(x) = \begin{cases} (x, 0), & \text{if} \quad x \leq 0, \\ (x, \frac{1}{n}x) & \text{if} \quad x > 0. \end{cases}$$

We can easily verify that $f_n \overset{\Gamma_C}{\to} f$ with $f(x) = (x, 0)$, $x \in R$. f_n and f satisfy all the conditions of Theorem 4.1. It is easy to verify that (S, f) is extended Hadamard well-posed with respect to $\{f_n\}$ and generalized Hadamard well-posed with respect to $\{f_n\}$.

Lemma 4.3 [5, Theorem 5.1] *Assume that* $f : S \to Y$ *and* $\varepsilon \geq 0$.

(i) $x_0 \in Eff(f, S, C, \varepsilon q)$ *implies* $x_0 \in Inf(\varphi_{x_0,\varepsilon} \circ f, S, \varepsilon)$*;*
(ii) $x_0 \in Inf(\varphi_{x_0,\varepsilon} \circ f, S, \varepsilon)$ *implies* $x_0 \in Eff(f, S, C, vq)$, $\forall v > \varepsilon$.

Similar to the proof of Theorem 4.1, we have the following Hadamard well-posedness properties corresponding to efficient points of vector-valued optimization problems.

Theorem 4.2 *Assume that* $f_n, f : S \to Y$, $f_n \xrightarrow{\Gamma_C} f$ *and* f *is strongly upper* C*-semicontinuous. Then*

(i) $\forall x_n \to \bar{x}$, $\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$, $\limsup_n val(S, \varphi_{x_n,\varepsilon_n} \circ f_n) \leq val(S, \varphi_{\bar{x},0} \circ f)$*, and for arbitrarily chosen* $\varepsilon \geq 0$, $\limsup_n val(S, \varphi_{x_n,\varepsilon} \circ f_n) \leq val(S, \varphi_{\bar{x},\varepsilon} \circ f)$*;*
(ii) $\exists \varepsilon_0 > 0$ *such that for all* $0 \leq \varepsilon \leq \varepsilon_0$, $Limsup_n[Eff(f_n, S, C, \varepsilon q)] \subset Eff(f, S, C, vq)$ *for every* $v > \varepsilon$*;*
(iii) $\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$, $Limsup_n[Eff(f_n, S, C, \varepsilon_n q)] \subset Eff(f, S, C, vq)$, $\forall v > 0$.

Proof The proofs of (i) and (ii) are similar to the proofs of Theorem 4.1(i) and (ii), respectively. We only need to prove (iii).

$\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$, let $\bar{x} \in \mathrm{Limsup}_n[\mathrm{Eff}(f_n, S, C, \varepsilon_n q)]$, i.e. $\exists \{n_k\} \subset N$, $x_{n_k} \in \mathrm{Eff}(f_{n_k}, S, C, \varepsilon_{n_k} q)$ such that $x_{n_k} \to \bar{x}$. From Lemma 4.3(i), $x_{n_k} \in \mathrm{Inf}(\varphi_{x_{n_k},\varepsilon_{n_k}} \circ f_{n_k}, S, \varepsilon_{n_k})$. Therefore, $\bar{x} \in \mathrm{Limsup}_{n_k}[\mathrm{Inf}(\varphi_{x_{n_k},\varepsilon_{n_k}} \circ f_{n_k}, S, \varepsilon_{n_k})]$. By Theorem 3.1(II), we have var-lim$\varphi_{x_{n_k},\varepsilon_{n_k}} \circ f_{n_k} = \varphi_{\bar{x},0} \circ f$. It can be deduced that

$$\mathrm{Limsup}_{n_k}[\mathrm{Inf}(\varphi_{x_{n_k},\varepsilon_{n_k}} \circ f_{n_k}, S, \varepsilon_{n_k})] \subset \mathrm{Inf}(\varphi_{\bar{x},0} \circ f, S).$$

Thus, $\bar{x} \in \mathrm{Inf}(\varphi_{\bar{x},0} \circ f, S)$. By Lemma 4.3(ii), we have that $\bar{x} \in \mathrm{Eff}(f, S, C, vq), \forall v > 0$. Hence, $\forall \varepsilon_n > 0$, $\varepsilon_n \to 0$, $\mathrm{Limsup}_n[\mathrm{Eff}(f_n, S, C, \varepsilon_n q)] \subset \mathrm{Eff}(f, S, C, vq), \forall v > 0$. □

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